We know from before that
\[ e^{j\theta} = \frac{A + D}{2} + j \left[ 1 - \left( \frac{A + D}{2} \right)^2 \right]^{1/2} = \cos \theta + j \sin \theta. \]

\[ \Rightarrow \theta = \cos^{-1}\left( \frac{A + D}{2} \right) = \cos^{-1}\left( 1 - \frac{d}{f_2} \right). \]

\[ f_2 = \frac{2}{3} \Rightarrow \theta = \cos^{-1}\left( 1 - \frac{2}{3} \right) = \cos^{-1}\left( -\frac{1}{2} \right) = \frac{2\pi}{3} (120^\circ). \]

Our solution for the position \( \gamma \) is given by

\[ \gamma = \gamma_{\text{max}} \sin \left( s \frac{2\pi}{3} + \alpha \right) \]

at \( s = 0 \)

\[ \gamma_{\text{max}} \sin \alpha = -\gamma_0 = \]

\[ \sin \alpha = -1. \quad \gamma_{\text{max}} = \gamma_0, \]

\[ \alpha = -\frac{\pi}{2} \]

\[ \gamma = \gamma_0 \sin \left( s \frac{2\pi}{3} - \frac{\pi}{2} \right), \quad (\text{location on } M_1) \]

If we want to know location at other minor \( M_2 \) we can use a different unit cell.
Solution \[ n = 2\gamma_0 \sin \left( p \frac{2\pi}{3} - \frac{\pi}{6} \right). \]

Different choices of unit cell \( \Rightarrow \) different solutions and information only about the position at the edge of that unit cell.

Repetitive Ray Paths.

- How many round trips before the ray returns to its original position?

\[ \gamma_0 = \gamma_{\text{max}} (s \theta + \alpha). \]
\[ \theta = \cos^{-1} \left( \frac{A-D}{2} \right). \]

For repetitive \( m \) round trips is required where
\[ m \theta = 2n\pi \quad \text{is some multiple of} \quad 2\pi. \]
\[ n < \frac{m}{2}. \]

\[ m = \frac{2\pi \eta}{\theta} \quad \text{for} \quad \theta = \frac{2\pi}{3} \quad m = 3\pi. \]

\[ n = 1; \quad m = 3 \quad \text{is a solution}. \]
Initial Conditions - Stable cavity.

If cavity is stable \( \Theta = \cos^{-1}\left(\frac{A+D}{Z}\right) \).
\[
\cos \Theta = \left(\frac{A+D}{Z}\right)
\]

Suppose initial conditions are \( r_0 \) and \( r_0' \)
\[
\gamma = |r_{\text{max}}| \sin \left(\Theta + \alpha\right).
\]
\[
s = 0
\]
\[
\gamma = |r_{\text{max}}| \sin \alpha = r_0.
\]

After 1 round trip:
\[
\gamma = A \cdot r_0 + B \cdot r_0' = |r_{\text{max}}| \sin \left(\Theta + \alpha\right).
\]
\[
A \cdot r_0 + B \cdot r_0' = |r_{\text{max}}| \sin \left(\Theta + \alpha\right).
\]
\[
= |r_{\text{max}}| (\sin \Theta \cos \alpha + \cos \Theta \sin \alpha)
\]
\[
= |r_{\text{max}}| \sin \Theta \cos \alpha + |r_{\text{max}}| \frac{A+D}{Z} \sin \alpha
\]
\[
|r_{\text{max}}| \cos \alpha = \frac{1}{\sin \Theta} \left[ r_0 \left(\frac{A-D}{Z}\right) + B \cdot r_0' \right]
\]

We have \( |r_{\text{max}}| \sin \alpha = r_0 \).

\[
\tan \alpha = \frac{r_0 \sin \left[ r_0 \left(\frac{A-D}{Z}\right) + B \cdot r_0' \right]}{1 - \left(\frac{(A+D)^2}{Z}\right)^{1/2}}
\]

but \( \sin \Theta = \left(1 - \left(\frac{A+D}{Z}\right)^2\right)^{1/2} \).

\[
\tan \alpha = \left[ \frac{r_0 \left(1 - \left(\frac{A-D}{Z}\right)^{1/2}\right)}{r_0 \left(\frac{A-D}{Z}\right) + B \cdot r_0'} \right]
\]

Once \( \alpha \) is determined
\[
|r_{\text{max}}| = \frac{r_0}{\sin \alpha}.
\]
\[ \tan \alpha = \left[ \frac{r_0 \left( 1 - (A+D)^2 \right)^{1/2}}{r_0 \left( A - D \right) + Br_0} \right] \]

**Review:** Stability Criteria

\[ g_{1,2} \leq 1 \quad g_{1,2} = 1 - \frac{d}{R_{1,2}} \]

Solution for stable system \( r = r_{\text{max}} \sin(\theta \theta + \alpha) \)

where \( \theta \) is the number of round trips.

**Unstable Cavities:**

Recall difference equation is of form

\[ r_{s+2} - 2 \left( \frac{A+D}{2} \right) r_{s+1} + r_s = 0 \]

For unstable cavities, solution is of the form

\[ r_s = r_0 (F)^s \]

both solutions are real.

\[ F_1,2 = \frac{A+D}{2} \pm \left[ \left( \frac{A+D}{2} \right)^2 - 1 \right]^{1/2} \]

General solution: \( r_s = r_0 (F_1)^s + r_0 (F_2)^s \)

Unstable cavities: \( \frac{A+D}{2} \geq 1 \) or \( -1 \geq \frac{A+D}{2} \)

Remember \( \cos \theta \) for stable cavities.

\[ \Rightarrow F_1 \text{ or } F_2 \text{ has a magnitude } > 1 \text{ and thus after a few round trips the solution is approximately } r_s \sim r_0 (F_2)^s \text{ or } \text{solution.} \]
Evaluation of $r_a$ & $r_b$.

\[ S = 0 \quad r_b = r_a + r_b = r_0 \]

\[ S = 1 \quad r_1 = r_a F_1 + r_b F_2 = A r_0 + B r_0' \]

Solving for $r_a$ and $r_b$ yields:

\[ r_b = \frac{1}{F_1 - F_2} \left( r_0 (F_1 - A) - B r_0' \right) \]

\[ r_a = \frac{1}{F_2 - F_1} \left( r_0 (F_2 - A) - B r_0' \right) \]

So far we have discussed ray tracing in homogeneous material systems. What happens in inhomogeneous, or lens-like media?

thickness causes changes to our simple modeled picture.

**For example.**

Lens – constant index – quadratic physical path.

\[ E_R(x, y) = E_L(x, y) e^{i k \frac{x^2 + y^2}{2 f}} \]

How do we get this?

Recall $v = \frac{c}{n}$.

\[ E \propto e^{i k z} \]

for right going wave

\[ k = \frac{2 \pi}{\lambda} = \frac{2 \pi n}{\lambda_0} \]

index causes phase difference

\[ \Delta = \frac{FB' - FC'}{FB'} = \frac{1}{\sqrt{r^2 + f^2}} \quad FC' = f \]

If the lens is not there $B'$ & $A'$ would be in phase, however $A'$ and $C'$ are in phase.
\[ \Delta = F'B' - FC' = \sqrt{r^2 + f^2} - f = f \sqrt{1 + \frac{r^2}{f^2}} - f. \]

\[ \sqrt{1 + \frac{r^2}{f^2}} \approx 1 + \frac{r^2}{2f^2} \]

if \( f \gg r \) (paraxial approximation).

\[ \Delta = f \left(1 + \frac{r^2}{2f^2}\right) - f = \frac{r^2}{2f} = \frac{x^2 + y^2}{2f} \]

Phase difference = \# \lambda x 2\pi = \frac{\text{path difference}}{\lambda} \times 2\pi = \Delta \times \frac{2\pi}{\lambda}

\[ \Rightarrow \ E_R(x, y) = E_L(x, y) e^{i k \Delta} = E_L(x, y) e^{i k (\frac{x^2 + y^2}{2f})}. \]

\[ \nu = \frac{c}{n} \] - wave inside glass is moving slower.

Another example.

\[ n(r) \sim r^2 \] (distance from center).

eg. \[ n(x, y) = n_0 \left[1 - \frac{k_2}{2k} (x^2 + y^2)\right] \] \( k_2 \) = constant of material

1. graded index fiber.

2. propagation of intense laser beam through a medium with \( n = n(I) \).

\[ n = n_0 + n_2 I \]

Gaussian spatial profile

\[ I(r) = I_0 e^{-2 \left(\frac{r}{\omega_0}\right)^2} \]

\( \omega_0 = \text{half width at } \frac{E_0}{e} \).
\[ I(r) = I_0 \left[ 1 - 2 \left( \frac{r^2}{w_0^2} \right) \right] \quad \text{for} \quad r < w_0. \]

\[ \Rightarrow n = n_0 + n_2 I_0 - \frac{2n_2 r^2}{\eta_0^2} \]  
where \( n_2 \) is a constant of the material.

\[ = n_0' - 2n_2 \left( \frac{x^2 + y^2}{w_0^2} \right) = n_0' \left[ 1 - \frac{2n_2}{n_0' w_0^2} (x^2 + y^2) \right]. \]

\( \Rightarrow \) Gaussian beam propagating in weakly absorbing medium (change index due to heating).

\[ \frac{dn}{dT} \neq 0 \]

\[ \frac{dn}{dT} > 0 \quad (n \text{ increases in center of beam relative to wings \Rightarrow converging lens \text{ "self focusing"}}) \]

\[ \frac{dn}{dT} < 0 \Rightarrow \text{diverging lens \text{ "self-defocusing"}}. \]

(3) Optical pumping of solid state lasers:

3 elements to laser:

- gain medium
- pumping mechanism
- cavity (resonator).

\[ \text{Gain medium: gas, organic dyes, some solid state materials} \]

\[ \text{Pumping: electrical discharge, chemical reactions, another laser, flash lamps.} \quad \text{Nd:YAG + flashlamps} \]

\[ \text{Ti: Sapphire + laser.} \]
Energy gain medium not converted to laser $\Rightarrow$ heat
$\Rightarrow$ Temperature gradient
$T(0) > T(r_l)$: Temp at center greater than at outside.

$\frac{dn}{dt} \neq 0 \Rightarrow$ lens effect.

General differential equation describes propagation through inhomogeneous medium.

Paraxial approximation:

\[ \Delta r \]

Snells' law of refraction:

\[ n_1 \sin (\frac{\pi}{2} - \theta_1) = n_2 \sin (\frac{\pi}{2} - \theta_2). \]

\[ n_1 \cos (\theta_1) = n_2 \cos (\theta_2). \]

So

\[ n(r) \cos (\theta_1) = n(r + \Delta r) \cos (\theta_1 + \Delta \theta). \]

Expand \( n(r) \) in Taylor series.

\[ n(r + \Delta r) \approx n(r) + \frac{dn}{dr} \Delta r + \cdots. \]

\[ \cos (\theta_1 + \Delta \theta) = \cos \theta_1 \cos \Delta \theta - \sin \theta_1 \sin \Delta \theta. \]

\[ n(r) \cos \theta_1 = [n(r) + \frac{dn}{dr} \Delta r] \left[ \cos \theta_1 \cos \Delta \theta - \sin \theta_1 \sin \Delta \theta \right]. \]

For small \( \Delta \theta \), \( \cos \Delta \theta \approx 1 \), \( \sin \Delta \theta \approx \Delta \theta. \]
\[ n(r) \cos \theta_1 = n(r) \cos \theta_1 - n(r) \sin \theta_1 (\Delta \theta) + \frac{\partial n}{\partial r} \Delta r (\cos \theta_1 - \Delta \theta \sin \theta_1). \]

\[ \Rightarrow n(r) \sin \theta_1 \Delta \theta = \frac{\partial n}{\partial r} \Delta r \cos \theta_1. \]

or \[ \frac{\partial n}{\partial r} = n(r) \tan \theta_1 \left( \frac{\Delta \theta}{\Delta r} \right). \]

but parallel approx: \[ \tan \theta_1 = \frac{\Delta r}{\Delta z}, \]

\[ \frac{\partial n}{\partial r} = n(r) \frac{\Delta r}{\Delta z} \frac{\Delta \theta}{\Delta r} = n(r) \frac{\Delta \theta}{\Delta z}. \]

\[ \theta_1 = \frac{\Delta r}{\Delta z} \Rightarrow \Delta \theta = \Delta \left( \frac{\Delta r}{\Delta z} \right) \Rightarrow \frac{\Delta \theta}{\Delta z} = \frac{\Delta}{\Delta z} \left( \frac{\Delta r}{\Delta z} \right) \]

as \( \Delta \to 0 \) \[ \Rightarrow \frac{\Delta \theta}{\Delta z} \to \frac{\partial^2 r}{\partial z^2}. \]

\[ \therefore \frac{\partial n}{\partial r} = n(r) \frac{\partial^2 r}{\partial z^2}, \]

- solve this equation to describe ray motion in inhomogeneous medium.

Ray Tracing - path only tells us the path

no amplitude, phase, spatial extent

\[ \rightarrow \text{Maxwell's equations} \rightarrow \text{Approx solar wave equation}. \]

- Isotropic, Homogeneous, non-conducting medium.

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t} \quad (1) \]

\[ \nabla \times \vec{H} = \frac{\partial \vec{E}}{\partial t} = e \frac{\partial \vec{E}}{\partial t} \quad (2) \]