Optics of Anisotropic Media

1. Totally random collections of atoms are isotropic: gases, liquids; amorphous solids are isotropic.

2. If molecules are anisotropic & orientated not totally random → medium is anisotropic.

3. Molecules are structured, e.g. crystals, in general medium is anisotropic.


Refractive indices — recall for anisotropic media:

\[ D_{ij} = \sum_j E_{ij} E_j \]  \[ i, j = x, y, z \] indicate \( x, y, z \).

- Material (dielectric) properties characterized by \( 3 \times 3 \) tensor of second rank called the electric permittivity tensor. \( \mathbf{\varepsilon} \)

\[ \mathbf{\varepsilon} = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix} \] \[ E_{ij} = E_{ji} \] (symmetry).

⇒ six independent numbers.

Principal Axes and Principal Refractive Index.

⇒ A coordinate system can always be found for which \( E_{ij} \) vanish.⇒

\[ D_1 = \varepsilon_x E_x \] \[ D_2 = \varepsilon_y E_y \] \[ D_3 = \varepsilon_z E_z \].

\[ E = E_x \mathbf{\hat{x}} \Rightarrow D = D \times \mathbf{\hat{x}} = \varepsilon_x E_x \mathbf{\hat{x}}. \] This is the principal axes and principal planes of the crystal.
\[ n_1 = \left( \frac{\varepsilon_1}{\varepsilon_0} \right)^{1/2}, \quad n_2 = \left( \frac{\varepsilon_2}{\varepsilon_0} \right)^{1/2}, \quad n_3 = \left( \frac{\varepsilon_3}{\varepsilon_0} \right)^{1/2}. \]

\( n_1, n_2, n_3 \): principal refractive indices
(\( \varepsilon_0 = \) permittivity of free space).

### Biaxial, Uniaxial, Isotropic Crystals

- **Isotropic**: \( n_1 = n_2 = n_3 \)
- **Uniaxial**: \( n_1 = n_2 \) and \( n_3 \) different.
- **Biaxial**: \( n_1, n_2, n_3 \) all different.

### Uniaxial
\( n_1 = n_2 = n_0, \quad n_3 = n_\infty \) 3 notation that is used.
- \( n_0 \): ordinary index
- \( n_\infty \): extraordinary index

\( n_\infty > n_0 \) (positive uniaxial).
\( n_\infty < n_0 \) (negative uniaxial).

- **z** axis in a uniaxial crystal is called the optic axis.

### Impermeability Tensor

\[
\bar{D} = \varepsilon\bar{E} \quad \Rightarrow \quad \bar{E} = \varepsilon^{-1}\bar{D}
\]

\[
\bar{\eta} = \varepsilon\varepsilon^{-1} : \text{Impermeability tensor.}
\]

\[
\varepsilon_0 \bar{E} = \bar{\eta}\bar{D}
\]

### Geometrical Representation of Vectors and Tensors:

**Vector**

\[
\bar{p} = x_1 \hat{x} + y_1 \hat{y} + z_1 \hat{z}.
\]

\[
\begin{bmatrix}
  x_1 \\
  y_1 \\
  z_1
\end{bmatrix}
\]

3 numbers represent vector.
9 numbers represent tensor.

Symmetrical tensor - 6 numbers. (ellipsoid) quadratic surface.

\[ \sum_{ij} \epsilon_{ij} x_i x_j = 1. \]

known as quadric representation.

In principal coordinate system: \( \epsilon_{ij} \) is diagonal:

\[ \epsilon_1 x_1^2 + \epsilon_2 x_2^2 + \epsilon_3 x_3^2 = 1. \]

Principal axes are those of the tensor, axes are half-lengths:

\[ \epsilon_1^{-1/2}, \epsilon_2^{-1/2}, \epsilon_3^{-1/2} \]

Index Ellipsoid (optical indicatrix)

\[ \eta = \epsilon_0 \varepsilon^{-1} \]

\[ \sum \eta_{ij} x_i x_j = 1. \]

→ Principal axes:

\[ \frac{x_1^2}{\eta_1^2} + \frac{x_2^2}{\eta_2^2} + \frac{x_3^2}{\eta_3^2} = 1. \]

\[ \eta = \begin{bmatrix} \frac{1}{\eta_1^2} & 0 & 0 \\ 0 & \frac{1}{\eta_2^2} & 0 \\ 0 & 0 & \frac{1}{\eta_3^2} \end{bmatrix} \]

\( x_1, y_1, z_1 \) are the principal axes.

\( \eta_1, \eta_2, \eta_3 \): refractive indices

isotropic: sphere uniaxial: \( \eta_1 = \eta_2 \)
Propagation along a principal axis.

Normal modes. \( x-y-z \) : coordinate system.

\[
\frac{c_0}{n_1} \quad \frac{c_0}{n_2}
\]

\( D_1 = E_1E_1 \quad C^2 = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \)

Normal modes for propagation along \( x \)-direction are \( x, y \) polarized waves.

Polarization along an arbitrary direction.

\( \hat{z} \), arbitrary polarization in \( x-y \) plane.

Analyse as sum of normal modes, \( \hat{x} \) polarized mode goes at \( \frac{c_0}{n_1} \), \( \hat{y} \) polarized at \( \frac{c_0}{n_2} \) \( \Rightarrow \) phase change.

\[
\phi_x = n_1kd \quad \phi_y = n_2kd.
\]

\( \phi = \phi_x - \phi_y = (n_2 - n_1)kd \Rightarrow \text{elliptically polarized light out.} \)

\( \Rightarrow \text{wave retarder.} \)
Propagation in Arbitrary Direction.

\( \mathbf{\hat{u}} \) - direction of propagation.

Normal modes are linearly polarized waves, \( n_a \) and \( n_b \) are the refractive indices and the direction of these modes are determined as follows. (See attached picture).

1. Draw plane passing through origin of the index ellipsoid normal to \( \mathbf{\hat{u}} \). The intersect of this plane with the ellipsoid is called the index ellipse.
2. Half-lengths of the index ellipse are \( n_a \) and \( n_b \) of the normal modes.
3. Directus of major and minor axes of the index ellipse are directions of the vectors \( \mathbf{\hat{D}_a} \) and \( \mathbf{\hat{D}_b} \) for the normal modes (these are orthogonal).
4. \( \mathbf{\hat{E}_a} \) and \( \mathbf{\hat{E}_b} \) are determined from \( \mathbf{\hat{D}_a} \) and \( \mathbf{\hat{D}_b} \)

\[
\mathbf{\hat{D}_a} = \mathbf{\hat{E}_a} \mathbf{\hat{E}_a} \quad \mathbf{\hat{D}_b} = \mathbf{\hat{E}_b} \mathbf{\hat{E}_b} \quad n_a = \left( \frac{E_a}{E_0} \right)^{\frac{1}{2}}
\]

\[
\text{The Dispersion Relation.} \quad \mathbf{\hat{D}} = \mathbf{\hat{E}} \mathbf{\hat{E}} \quad \text{All fields: exp}(\text{-j}k \mathbf{\hat{u}} \cdot \mathbf{\hat{x}}) \quad \mathbf{\hat{x}} = k \mathbf{\hat{u}}
\]

\[
\rightarrow \mathbf{\hat{k}} \times \mathbf{\hat{H}} = -\omega \mathbf{\hat{D}} \quad (\mathbf{\hat{D}} \text{ Not necessarily parallel to } \mathbf{\hat{E}}).
\]

\[
\mathbf{\hat{k}} \times \mathbf{\hat{E}} = \omega \mu_0 \mathbf{\hat{H}}.
\]

(see picture on page 215)

\( \mathbf{\hat{D}} \) normal to both \( \mathbf{\hat{k}} \) & \( \mathbf{\hat{H}} \)

\( \mathbf{\hat{H}} \) normal to both \( \mathbf{\hat{k}} \) & \( \mathbf{\hat{E}} \)

\[
\mathbf{\hat{S}} = \frac{1}{2} \mathbf{\hat{E}} \times \mathbf{\hat{H}}^* \quad \text{(direct of power flux) orthogonal to } \mathbf{\hat{E}} \text{ & } \mathbf{\hat{H}}.
\]
The Dispersion Relation

To determine the normal modes for a plane wave traveling in the direction \( \hat{u} \), we use Maxwell's equations (5.3-2) to (5.3-5) and the medium equation \( D = \varepsilon E \). Since all fields are assumed to vary with the position \( r \) as \( \exp(-j\mathbf{k} \cdot \mathbf{r}) \), where \( \mathbf{k} = k \hat{u} \), Maxwell's equations (5.3-2) and (5.3-3) reduce to

\[
\mathbf{k} \times \mathbf{H} = -\omega \mathbf{D} \tag{6.3-8}
\]

\[
\mathbf{k} \times \mathbf{E} = \omega \mu_0 \mathbf{H} \tag{6.3-9}
\]

It follows from (6.3-8) that \( \mathbf{D} \) is normal to both \( \mathbf{k} \) and \( \mathbf{H} \). Equation (6.3-9) similarly indicates that \( \mathbf{H} \) is normal to both \( \mathbf{k} \) and \( \mathbf{E} \). These geometrical conditions are illustrated in Fig. 6.3-7, which also shows the Poynting vector \( \mathbf{S} = \frac{1}{2} \mathbf{E} \times \mathbf{H}^* \) (direction of power

\[\text{Figure 6.3-7} \quad \text{The vectors } \mathbf{D}, \mathbf{E}, \mathbf{k}, \text{ and } \mathbf{S} \text{ all lie in one plane to which } \mathbf{H} \text{ and } \mathbf{B} \text{ are normal.} \]

\( \mathbf{D} \perp \mathbf{k} \) and \( \mathbf{E} \perp \mathbf{S} \).
\( \vec{D}, \vec{E}, \vec{K} \) and \( \vec{S} \) lie in one plane to which \( \vec{H} \) and \( \vec{B} \) are normal.

In this plane \( \vec{D} \perp \vec{K} \) and \( \vec{S} \perp \vec{E} \). \( \vec{D} \) is not necessarily parallel to \( \vec{E} \), \( \vec{S} \) is not necessarily parallel to \( \vec{K} \).

\[ \vec{E} \times (\vec{E} \times \vec{E}) + \omega^2 \mu_0 \vec{E} \cdot \vec{E} = 0. \]

Vector equation, \( \vec{E} \) must satisfy. In principal axes:

\[ \begin{bmatrix} 2n_1^2k_0^2 - k_2^2 - k_3^2 & k_1k_2 & k_1k_3 \\ k_1k_2 & n_2^2k_0^2 - k_1^2 - k_3^2 & k_2k_3 \\ k_1k_3 & k_2k_3 & n_3^2k_0^2 - k_1^2 - k_2^2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

A

\[ \vec{K} = (k_1, k_2, k_3), \quad k_0 = \frac{\omega}{c_0} \quad (n_1, n_2, n_3) - \text{refractive indices}. \]

Nontrivial solution if determinant \( A = 0 \).

\[ \Rightarrow \omega = \omega(k_1, k_2, k_3) \quad \text{(nonlinear function).} \]

\( \omega(k_1, k_2, k_3) \) = dispersion relation is the equation of a surface \( k_1, k_2, k_3 \) known as the normal surface or \( \vec{K} \) surface.

Intersection of \( \vec{K} \) with \( K \) surface determines the vector \( \vec{K} \) with \( k = n \omega / c_0 \).

\[ \Rightarrow \text{index of refraction}. \]

Two intersecting \( \Rightarrow \) two normal modes of each \( \vec{K} \).

Example. Special Case: Uniaxial Crystals. \( \eta_1 = \eta_2 = \eta_0 \), \( \eta_3 = \eta_e \).

Index ellipsoid is an ellipsoid of revolution.

\[ \begin{array}{c}
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\text{Example: Special Case: Uniaxial Crystals.} \\
\eta_1 = \eta_2 = \eta_0, \quad \eta_3 = \eta_e. \\
\text{Index ellipsoid is an ellipsoid of revolution.}
\end{array}
\end{array} \]
Equation for ellipse: \( \frac{x^2}{n_0^2} + \frac{z^2}{n_e^2} = 1 \)

\[ z = n_e(\theta) \sin \Theta \quad x = n_e(\theta) \cos \Theta. \]

\[ \Rightarrow \frac{\sin^2 \Theta}{n_e^2} + \frac{\cos^2 \Theta}{n_0^2} = \frac{1}{n_e^2(\theta)}. \]

\( n_a = n_0 \quad n_b = n(\theta) \).

First is called ordinary ray. Second is called extraordinary ray.

\( n_e(\theta) \) - refractive index of the extraordinary ray.

\[ n_0 \leq n_e(\theta) \leq n_e \quad \text{if } n_e > n_0 \]

\[ n_0 \geq n_e(\theta) \geq n_e \quad \text{if } n_0 > n_e. \]
Answers to Questions:

1. Does phase change ⇒ frequency change?
2. What determines color, λ or f?
3. \( \mathbf{E} = k \hat{z} \) How does this affect \( \mathbf{E_x} \) & \( \mathbf{E_y} \)?

Consider the following:

\[
\begin{align*}
\mathbf{E} &= \mathbf{E_x} \hat{x} \quad \mathbf{k} = k \hat{z} \\
\text{In an isotropic medium} \\
D_x &= E_x E_x \\
E_x &= \varepsilon_0 n_x^2 \\
or \quad n_x &= \left( \frac{E_x}{\varepsilon_0} \right)^{1/2}
\end{align*}
\]

Maxwell's Equations ⇒ Wave Equation.

Assume time harmonic:

\[
\mathbf{E} = \mathbf{E_0} e^{j \omega t} \\
\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}
\]

⇒ where \( \mathbf{E} = \mathbf{E_x} \hat{x} \quad \mathbf{D} = E_x E_x \hat{x} \)

\[
\nabla \times \mathbf{H} = j \omega \mathbf{D} \\
\nabla \times \mathbf{E} = -j \omega \mathbf{B} \\
\n\nabla \cdot \mathbf{D} = 0 \\
\n\nabla \cdot \mathbf{B} = 0
\]

\[
\nabla \times \nabla \times \mathbf{E} = -j \omega (\nabla \times \mathbf{B}) = -\nabla^2 \mathbf{E} + \nabla (\nabla \cdot \mathbf{E})
\]

\[
\nabla \cdot \mathbf{E} = 0 \quad (\text{nondispersive, homogeneous})
\]

\[
\nabla^2 \mathbf{E} = -j \omega (j \omega \mu) \mathbf{E} = \omega^2 \mu \mathbf{E}
\]
\[ \nabla^2 \vec{E} = \omega^2 \mu \vec{D}, \]  

general Equat.  

\[ \vec{D} = E_x \hat{x} = E_x \vec{E}_x(\vec{r}), \quad \nabla^2 \vec{E} = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \vec{E}_x(\vec{r}). \]  

\[ \nabla^2 \vec{E}_x(\vec{r}) = \omega^2 \mu \varepsilon_\infty \vec{E}_x(\vec{r}) = 0. \]

\[ \omega^2 \mu \varepsilon_\infty = \frac{\omega^2 \varepsilon_\infty^2}{C_0^2} \quad \quad C_0 = \frac{1}{\sqrt{\varepsilon_\infty \mu_0}} \quad \quad \varepsilon_\infty = \varepsilon_0 \eta_x^2 \]

3. Note it is \( \eta_x \) - the index in the \( x \)-direction.

Now, for plane waves:  

\[ \vec{E}(\vec{r}) = \vec{E}_0 \exp(-j k \vec{r} \cdot \vec{r}). \]

\[ \Rightarrow \vec{E}_x(\vec{r}) = \frac{\vec{E}_0}{\varepsilon} \exp(-j k z \hat{z}). \]

\( \varepsilon \) = spatial dependence.

\( \vec{E}_x(\vec{r}) \) is constant vector.

\[ \Rightarrow \nabla^2 \vec{E} = \nabla^2 \vec{E}_x(\vec{r}) = k_z^2 \vec{E}_0 \exp(-j k z \hat{z}). \]

\[ \therefore \quad k_z^2 \vec{E}_0 \exp(-j k z \hat{z}) - \omega^2 \varepsilon_\infty \vec{E}_x \exp(-j k z \hat{z}) \hat{x} = 0. \]

\[ \Rightarrow \quad \frac{k_z}{C_0} = \frac{\omega \eta_x}{C_0}. \]

What if \( \vec{E} = (E_x^0 \exp(-j k_x z \hat{x}) + E_y^0 \exp(-j k_y z \hat{y}) \). \( \vec{D} = E_x \hat{x} + E_y \hat{y} \)

we would get

\[ \exp(-j k z \hat{z}) \left( k_x^2 E_x^0 \hat{x} + k_y^2 E_y^0 \hat{y} \right). \]

\[ \Rightarrow \left( k_x^2 \hat{x} + k_y^2 \hat{y} \right) - \omega^2 \frac{\eta_x^2}{C_0^2} \hat{x} - \omega^2 \frac{\eta_y^2}{C_0^2} \hat{y} = 0 \]

\[ k_x = \frac{\omega \eta_x}{C_0} \quad k_y = \frac{\omega \eta_y}{C_0}. \]
Because of the anisotropic behavior of the medium, we have two different values $\eta_x$, $\eta_y$ in the $x$-direction and the $y$-direction.

$$k = \frac{2\pi n}{\lambda_0} \quad \Rightarrow \quad \lambda = \frac{\lambda_0}{\eta}$$

$$\eta_x = \frac{\lambda_0}{n_x}, \quad \eta_y = \frac{\lambda_0}{n_y}$$

\[ \Delta \Phi = \Phi_y - \Phi_x = (\eta_y - \eta_x) k_0 d. \]

Solution: \[ \underline{E} = E_x \cos(\omega t - k_{ex} z) \hat{x} + E_y \cos(\omega t - k_{ey} z) \hat{y} \]

Polarization: \[
\begin{bmatrix}
E_x^0 \\
E_y^0 e^{i\Delta \Phi}
\end{bmatrix}
\]

Phase velocity: \[ \omega t - k_{ex} z = \text{constant phase.} \]

\[
\frac{d}{dt}(\omega t - k_{ex} z) = 0 \quad \Rightarrow \quad \omega - k_{ex} \frac{dz}{dt} = 0.
\]

Or \[ \frac{dz}{dt} = \frac{\omega}{k_{ex}} \]

\[ k_{ex} = \frac{2\pi n_x}{\lambda_0} \]

\[ k_{ey} = \frac{2\pi n_y}{\lambda_0} \quad \Rightarrow \quad \text{different phase velocities.} \]

2. Color determined by frequency, $\omega$.

In vacuum is generally quoted for wavelengths of color, not that $\lambda$ changes but frequency $\omega$ stays the same.
\[ \cos \left( \frac{\omega t - 2\pi n x}{\lambda_0} \right) \cos \left( \frac{\omega t - 2\pi n y}{\lambda_0} \right) \]

1. At \( z = z_0 \) (a constant) the amplitude of both cos terms change with \( \omega \).