EE 311 Lecture 7

Students should review Section 7.1 on Poles, Zeros, & Bode Plots and Appendix F on Single Time Constant (STC) Circuits.

7.2 The Amplifier Transfer Function $A(s)$

$$A(s) = V_o(s) / V_i(s)$$

Fig. 7.5

$|A| \text{ dB}$

0 $\rightarrow \omega_H$

$\omega_L \rightarrow \omega_H$

$|A| \text{ dB}$

Capacitively Coupled Amp.
Lower Half-Power Freq. \( \omega_L = 2\pi f_L \)

Can be estimated using Method of Short-Circuit Time Constants

Upper Half-Power Freq. \( \omega_H = 2\pi f_H \)

Can be estimated using Method of Open-Circuit Time Constants

Gray & Seerle "Electronic Principles"

John Wiley 1969
Consider the low-frequency equivalent circuit of an electronic circuit with 3 coupling and/or bypass capacitors: C1, C2, & C3.

\[ I_1' = I_1 = g_{11} V_1 + g_{12} V_2 + g_{13} V_3 \\
I_2 = g_{21} V_1 + g_{22} V_2 + g_{23} V_3 \\
I_3 = g_{31} V_1 + g_{32} V_2 + g_{33} V_3 \]

Network Inside Box Is Purely Resistive

\[ 1/R_{15} = g_{11} = \frac{I_1}{V_1} \text{ with } V_2 = 0 \text{ and } V_3 = 0 \]
\[ g_{11} = \text{Input Conductance with } V_2 \text{ and } V_3 \text{ shorted} \]

\[ 1/R_{25} = g_{22} = \frac{I_2}{V_2} \text{ with } V_1 = 0 \text{ and } V_3 = 0 \]
\[ g_{22} = \text{Input Conductance with } V_1 \text{ and } V_3 \text{ shorted} \]

\[ 1/R_{35} = g_{33} = \frac{I_3}{V_3} \text{ with } V_1 = 0 \text{ and } V_2 = 0 \]
\[ g_{33} = \text{Input Conductance with } V_1 \text{ and } V_2 \text{ shorted} \]
The Method of Short-Circuit Time Constants - 2

To determine $g_{11}$
Remove $C_1$ and short out $C_2$ & $C_3$ and calculate the conductance ratio $I_1 / V_1$

To determine $g_{22}$
Remove $C_2$ and short out $C_1$ & $C_3$ and calculate the conductance ratio $I_2 / V_2$

To determine $g_{33}$
Remove $C_3$ and short out $C_1$ & $C_2$ and calculate the conductance ratio $I_3 / V_3$

LowFreq. Equiv. Cht Including $C_1$, $C_2$ & $C_3$

\[ I_1' = (g_{11} + sC_1)V_1 + g_{12}V_2 + g_{13}V_3 \]
\[ I_2' = g_{21}V_1 + (g_{22} + sC_2)V_2 + g_{23}V_3 \]
\[ I_3' = g_{31}V_1 + g_{32}V_2 + (g_{33} + sC_3)V_3 \]

The poles are determined by setting the determinant of the admittance matrix equal to zero.

\[ \Delta_y = \begin{vmatrix} g_{11} + sC_1 & g_{12} & g_{13} \\ g_{21} & g_{22} + sC_2 & g_{23} \\ g_{31} & g_{32} & g_{33} + sC_3 \end{vmatrix} = 0 \]
The term \( q_5 \) results from the sum of 3 terms each of which contains one capacitor \((CC, CC, CC)\) multiplied by the corresponding element in the conductance matrix. \( \Delta \) of \( q_5 \) is with \( s = 0 \) and \( \Delta y(s=0) = \Delta y \).
The Method of Short-Circuit Time Constants - Y

\[ \Delta y = \begin{vmatrix} (g_{11} + sC1) & g_{12} & g_{13} \\ g_{21} & (g_{22} + sC2) & g_{23} \\ g_{31} & g_{32} & (g_{33} + sC3) \end{vmatrix} \]

\[ \Delta y = (g_{11} + sC1) \begin{vmatrix} g_{22} & g_{23} \\ g_{32} & (g_{33} + sC3) \end{vmatrix} - (g_{21}) \begin{vmatrix} g_{12} & g_{13} \\ g_{32} & (g_{33} + sC3) \end{vmatrix} + (g_{31}) \begin{vmatrix} g_{12} & g_{13} \\ g_{22} & g_{23} \end{vmatrix} \]

\[ \Delta y = (g_{11} + sC1)[(g_{22} + sC2)(g_{33} + sC3) - g_{32}g_{23}] \]
\[ - (g_{21})[g_{12}(g_{33} + sC3) - g_{32}g_{13}] \]
\[ + (g_{31})[g_{12}g_{23} - g_{13}(g_{22} + sC2)] \]

\[ \Delta y = a_0 + a_1 s + a_2 s^2 + a_3 s^3 \]

\[ a_0: \text{ The term } a_0 \text{ can be determined by setting } s = 0 \text{ in both expression} \]

\[ a_0 = \Delta y (s=0) = g_{11}[g_{22}g_{33} - g_{32}g_{23}] \]
\[ - g_{21}[g_{12}g_{33} - g_{32}g_{22}] \]
\[ + g_{31}[g_{12}g_{23} - g_{13}g_{22}] \]
\[ = \Delta g = \text{Determinant of } g \text{ Matrix} \]
\[ \Delta g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = g_{11} \Delta g_{11} - g_{21} \Delta g_{21} + g_{31} \Delta g_{31} \]

\[ \Delta g = g_{11} \begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{vmatrix} - g_{21} \begin{vmatrix} g_{12} & g_{13} \\ g_{32} & g_{33} \end{vmatrix} + g_{31} \begin{vmatrix} g_{12} & g_{13} \\ g_{22} & g_{23} \end{vmatrix} \]

\[ \Delta y = \Delta g \]

\[ \Delta y = + \Delta g_{11} \Delta g_{11} + \Delta g_{22} \Delta g_{22} + \Delta g_{33} \Delta g_{33} \]

\[ + s \left[ C1 (g_{22} g_{33} - g_{32} g_{23}) + C2 (g_{11} g_{33} - g_{31} g_{13}) \right. \]

\[ + C3 (g_{11} g_{22} - g_{21} g_{12}) \left. \right] \]

\[ + s^2 \left[ C1 C2 g_{33} + C2 C3 g_{11} + C3 C1 g_{22} \right] \]

\[ + s^3 C1 C2 C3 \]

\[ \Delta y = \Delta g + s \left[ C1 \Delta g_{11} + C2 \Delta g_{22} + C3 \Delta g_{33} \right] \]

\[ + s^2 \left[ g_{11} C2 C3 + g_{22} C3 C1 + g_{33} C1 C2 \right] \]

\[ + s^3 C1 C2 C3 \]

\[ a_0 = \Delta g_{11} = \Delta g \]

\[ a_1 = C1 \Delta g_{11} + C2 \Delta g_{22} + C3 \Delta g_{33} \]

\[ a_2 = g_{11} C2 C3 + g_{22} C3 C1 + g_{33} C1 C2 \]

\[ a_3 = C1 C2 C3 \]
The Method of Short-Circuit Time Constants

The ratio $\frac{a_2}{a_3}$

$$\frac{a_2}{a_3} = \frac{g_{11}}{C_1} + \frac{g_{22}}{C_2} + \frac{g_{33}}{C_3} = \frac{1}{R_{15}C_1} + \frac{1}{R_{25}C_2} + \frac{1}{R_{35}C_3}$$

Remember: $R_{15} =$ Resistance facing $C_1$ with $C_2$ & $C_3$ shunted
$R_{25} =$ "  $C_2$ & $C_1$ & $C_3$ "
$R_{35} =$ "  $C_3$ & $C_1$ & $C_2$ "

The transfer function $\frac{V_o(s)}{V_{sig}(s)}$

We expect the transfer function $\frac{V_o(s)}{V_{sig}(s)}$ to have 3 poles and 3 zeros and to be of the form:

$$\frac{V_o(s)}{V_{sig}(s)} = \frac{b_0 + b_1 s + b_2 s^2 + b_3 s^3}{a_0 + a_1 s + a_2 s^2 + a_3 s^3}$$

Dividing numerator and denominator by $s^3$ yields

$$\frac{V_o(s)}{V_{sig}(s)} = \frac{b_0 / s^3 + b_1 / s^2 + b_2 / s + b_3}{a_0 + a_1 s + a_2 s^2 + a_3 s^3}$$

At mid-band: $s = j\omega$ is large and $1/s^3$, $1/s^2$, & $1/s$ are small

$$\left[\frac{V_o(j\omega)}{V_{sig}(j\omega)}\right]_{\text{midband}} = \left|\frac{b_3}{a_3}\right| = A_0$$

As $\omega$ decreases from midband, we expect the transfer function to have a dominant pole which causes the ratio $\left|\frac{V_o}{V_{sig}}\right|$ to decrease. The term $a_2/s = a_2/j\omega$ becomes important.

Typical Pole-Zero Plot

Plot $\left|\frac{V_o}{V_{sig}}\right|$ vs $\omega$

- $s = 0$
- $s = j\omega$
- $s =$ Double Zero
The Method of Short-Circuit Time Constants -

Just Below Mid-Band

\[ \frac{V_o(s)}{V_{sig}(s)} = \frac{b_3}{a_2/s + a_3} = \frac{b_3/a_3}{a_2 \frac{1}{s} + 1} \]

Setting \( s = j\omega \), \( b_3/a_3 = A_0 \), and \( a_2/a_3 = \omega L \)

\[ \frac{V_o(j\omega)}{V_{sig}(j\omega)} = \frac{A_0}{\omega L \frac{j\omega}{j\omega} + 1} \]

At \( \omega = \omega_L \)

\[ \frac{V_o(j\omega_L)}{V_{sig}(j\omega_L)} = \frac{A_0}{\frac{1}{j} + 1} = \frac{A_0}{-j + 1} \]

\[ \left| \frac{V_o(j\omega_L)}{V_{sig}(j\omega_L)} \right| = \left| A_0 \right| \frac{1}{V_Z} = 0.707 \left| A_0 \right| \]

\[ \left| \frac{V_o(j\omega_L)}{V_{sig}(j\omega_L)} \right|^2 = \frac{\left| A_0 \right|^2}{2} \]

Thus \( \omega_L \) is the lower half-power freq in rad/s.

\[ \omega_L \approx \frac{a_2}{a_3} \approx \frac{1}{R_{1s} C_1} + \frac{1}{R_{2s} C_2} + \frac{1}{R_{3s} C_3} \]

Four or More Capacitors in Low Freq Eq Ch +

\[ \omega_L \approx \frac{a_{n-1}}{a_n} = \sum_{j=1}^{n} \frac{1}{R_{is} C_j} \]
Fig. 5.34 Small-signal models for the MOSFET: (a) neglecting the dependence of $i_D$ on $v_{DS}$ in saturation (channel-length modulation effect); and (b) including the effect of channel-length modulation, modeled by output resistance $r_o = \frac{|V_A|}{I_D}$.

Fig. 5.37 The T model of the MOSFET augmented with the drain-to-source resistance $r_o$. 
Fig. 7.10  The classical capacitively coupled common-source amplifier.

\[ \omega_L = \frac{1}{R_{cis} C_{c1}} + \frac{1}{R_{s2s} C_{c2}} + \frac{1}{R_{c22s} C_s} = \frac{2\pi f_L}{\omega_L} \]

We need to calculate \( R_{cis} \), \( R_{s2s} \), & \( R_{c22s} \)

Fig. 7.11  The amplifier circuit of Fig. 7.10 prepared for finding the gain at low frequencies. The resistance \( 1/g_m \) shown is the FET internal resistance between gate and source looking into the source (i.e., that of the T model).
Fig. 2.10 The classical capacitively coupled common-source amplifier.

Calculation of $R_{cis}$

Set $V_i = 0$ and Short $C_{c2}$ & $C_s$

Replace $C_{c1}$ by $V_T$ & Calculate $I_T$; $R_{cis} = \frac{V_T}{I_T}$

$$R_{cis} = \frac{V_T}{I_T} = R + R_{01} \parallel R_{02}$$
Fig. 7.10  The classical capacitively coupled common-source amplifier.

Calculation of $R_{c2s}$
Set $V_i = 0$ and Short $C_{c1}$ & $C_s$
Replace $C_{c2}$ by $V_t$ & calculate $I_t$; $R_{c2s} = \frac{V_t}{I_t}$

$g_m V_{gs} = 0 \Rightarrow \text{open-circuit}$

$R_{c2s} = \frac{V_t}{I_t} = R_L + R_D || r_0$
Fig. Z10  The classical capacitively coupled common-source amplifier.

**Calculation of $R_{\text{CSS}}$**

Set $V_i = 0$ and short $C_{c1}$ & $C_{c2}$

Replace $C_s$ by $V_t$ & calculate $I_t$; $R_{\text{CSS}} = \frac{V_t}{I_t}$

$g_m V_{gs} = g_m [V_i - V_s] = g_m [0 - V_s] = -g_m V_s$

**Simplest Case** $g_0 = \infty$ (we omit $g_0$)

$I_t = g_m V_s + \frac{V_s}{R_s} = V_t \left[ \frac{g_m}{g_m + 1/R_s} \right] = V_t \left[ \frac{1}{g_m} + \frac{1}{R_s} \right]$

$R_{\text{CSS}} = \frac{1}{I_t} = \frac{1}{g_m + 1/R_s} = \frac{R_s}{1 + g_m R_s} = \frac{1}{g_m} / R_s$
Fig. 7.12 The output equivalent circuit (at low frequencies) for the amplifier in Figs. 7.10 and 7.11.

BJT Amplifier (Fig. 7.13)
Method of Short-Circuit Time Constants

\[ \omega_L = \frac{1}{R_C I_S C_{c1}} + \frac{1}{R_C E S C_{c2}} + \frac{1}{R_C E S C_E} \]

Fig. 7.13 The classical common-emitter amplifier stage. (The nodes are numbered for the purposes of the SPICE simulation in Example 7.9.)
Calculation of $R_{cis} \equiv \frac{V_t}{I_t}$

Set $V_s = 0$ and short $C_E \& C_{c2}$

Replace $C_{c1}$ by $V_t$ ($R_{cis} = \frac{V_t}{I_t}$)

$$\frac{V_t}{I_t} = R_{cis} = R_s + R_B \left\| \left[ r_x + r_m \right] \right\|$$
Fig. 7.14 Equivalent circuit for the amplifier of Fig. 7.13 in the low-frequency band.

Calculation of $R_{C2S}$

Set $V_s = 0$ and Short $C_{C1}$ and $C_E$

Replace $C_{C2}$ by $V_t$ ($R_{C2S} = V_t/I_t$)

No Signal Present ($I_b = 0$) so $V_{re} = 0$ and $g_m V_{re} = 0$

$R_{C2S} = \frac{V_t}{I_t} = R_L + R_{C}||r_o$
Fig. 7.14 Equivalent circuit for the amplifier of Fig. 7.13 in the low-frequency band.

Calculation of \( R_{ces} \)

Set \( V_S = 0 \) and Short \( C_{c1} \) \& \( C_{c2} \)

Replace \( C_E \) by \( V_t \) (\( R_{ces} = \frac{V_t}{I_t} \))

Simple case (Neglect \( r_o \))

\[ V_e = V_t \quad I_t = I_b' + \beta I_b \]

\[ V_t = \frac{V_e}{r_{\pi} + r_x + R_{B1}||R_{B2}} \quad I_t = (1+\beta) \frac{V_t}{r_{\pi} + r_x + R_{B1}||R_{B2}} \]

\[ \frac{V_t}{I_t} = R_{cess} = \frac{r_{\pi} + r_x + R_{B1}||R_{B2}}{(\beta + 1)} \]
For the Wilson current mirror of Fig. 6.20, replace the diode-connected transistor \( Q_1 \) with its incremental resistance \( r_e \), and replace \( Q_2 \) and \( Q_3 \) with their low-frequency hybrid-\( \pi \) models including \( r_e \) but excluding \( r_a \) (for simplicity). Note that all three transistors operate at equal dc currents and thus have identical model parameters. Apply a test voltage \( v_x \) between the collector of \( Q_1 \) and ground and determine the current \( i_x \) drawn from the source \( v_x \). From this, show that the output resistance \( R_o = v_x / i_x = \beta r_e / 2 \).

Q1 & Q2 Form Mirror

\[
\sqrt{V_{BE2}} = V_{BE1} \\
I_{E2} = I_{E1} \Rightarrow \beta_{e2} = \beta_{e1} = \beta_e \\
I_{B2} = I_{B1} \quad \beta_{p2} = \beta_{p1} = \beta_p \\
I_{C2} = I_{C1} \quad g_{m2} = g_{m1} = g_m
\]

\[ R_o = V_x / I_x \]
\[ I_X = I_{C3} = I_{R03} + 9m V_{\Pi3} = I_{R03} - 9m V_{\Pi3}' \]

\[ I_X = I_1 + I_2 \]

\[ I_1 = \frac{V_{E3}}{R_{E1}} + \frac{V_{E3}}{R_{\Pi2}} = \frac{V_{E3}}{R_{E1}} \frac{1}{(B_2 + 1)R_{E2}} \]

\[(B_2 + 1)R_{E2} \gg R_{E1} \Rightarrow I_1 \approx \frac{V_{E3}}{R_{E1}} \]

\[ I_2 = 9m_2 V_{\Pi2} + \frac{V_{E2}}{V_{R02}} = 9m_2 V_{E3} + 9_2 V_{E2} \]

Now we can show that \( 9_2 V_{E2} \ll 9m_2 V_{E3} \)

\( 9_2 V_{E2} < 9_2 V_{E3} \) because \( V_{E3} > V_{E2} + I_2 R_{\Pi2} \)

and \( I_2 R_{\Pi2} > 0 \). Thus \( 9_2 V_{E2} < 9_2 V_{E3} \ll 9m_2 V_{E3} \)

Since \( 9m_2 \gg 9_2 V_{E2} / V_{R02} \Rightarrow I_2 = 9m_2 V_{E3} \) (We can delete \( 9_2 \))

\[ R_{E1} = \frac{V_T}{I_{E1}} \quad 9m_2 = \frac{I_{C2}}{V_T} = \frac{\alpha_2 V_{E2}}{V_T} = \frac{\alpha_2}{R_{E2}} \]

\[ I_1 \quad \frac{I_1}{I_2} = \frac{V_{E3}/R_{E1}}{9m_2 / V_{E3}} = \frac{1}{9m_2 R_{E1}} = \frac{1}{\alpha_2 R_{E1}} \approx 1 \]

Thus \( I_1 = I_2 = \frac{I_X}{2} \)

Note that \( I_X \) splits into two parts \( I_1 \) and \( I_2 \)

Now calculate \( V_X = I_{R03} R_{R03} + V_{E3} \)

\[ x \rightarrow I_{R03} = I_X + 9m_3 V_{\Pi3}' \]

\[ V_{\Pi3}' = R_{\Pi3} \cdot I_2 = \frac{R_{\Pi3} I_X}{2} \]

The voltage drop across \( R_{\Pi3} \) is \( V_{\Pi3} = R_{\Pi3} I_2 = R_{\Pi3} \cdot \frac{I_X}{2} \)
\[ \Gamma_{03} = \Gamma_x + g_m v_3 \Gamma_{\Pi3} \Gamma_x/2 = \Gamma_x \left[ 1 + \frac{\beta_3}{2} \right] \]

\[ V_x = \Gamma_x \left[ 1 + \frac{\beta_3}{2} \right] + V_{e3} \]

\[ V_{e3} = \Gamma_1 x Re \left| \text{Re} \right| \text{Re} = \Gamma_1 x/2 \]

\[ V_x = \Gamma_x \Gamma_{03} \left[ 1 + \frac{\beta_3}{2} \right] + \Gamma_x \frac{Re_1}{2} \]

\[ R_0 = \frac{V_x}{\Gamma_x} = \Gamma_{03} \left[ 1 + \frac{\beta_3}{2} \right] + \frac{Re_1}{2} \]

\[ R_0 \approx \Gamma_{03} \frac{\beta_3}{2} = \frac{\beta R_0}{2} \]

Note that factor \( \frac{1}{2} \) appears because \( \Gamma_x \) divides more or less equally into two equal components \( \Gamma_1 \) and \( \Gamma_2 \). The voltage drop \( \text{Re} \) across \( \Gamma_{\Pi3} \) is \( \Gamma_2 \Gamma_{\Pi3} = \Gamma_x \Gamma_{\Pi3}/2 \) and not \( \Gamma_x \Gamma_{\Pi3} \) as obtained in a previous case. (See cascode section.)