Optics of Anisotropic media.

① Totally random collections of atoms are isotropic: gases, liquids; amorphous solids are isotropic

② If molecules are anisotropic & orientations not totally random
$\Rightarrow$ medium is anisotropic.

③ Molecules are structured, eg. crystals, in general medium is anisotropic.

④ Polycrystalline materials - generally isotropic.

Refractive indices. recall for anisotropic media:

$$D_i = \sum_j \epsilon_{ij} E_j \qquad i, j = 1, 2, 3 \text{ indicate } x, y, z.$$

$\Rightarrow$ material (dielectric) properties characterized by 3x3 tensor of second rank called the electric permittivity tensor. $\overleftrightarrow{\epsilon}$

$$\vec{D} = \overleftrightarrow{\epsilon}\,\vec{E} \qquad \epsilon_{ij} = \epsilon_{ji} \quad \text{(symmetry)}.$$

$\Rightarrow$ six independent numbers.

Principal axes and Principal Refractive Index.

$\Rightarrow$ A coordinate system can always be found for which ϵ_{ji} vanish. $\Rightarrow$

$$D_1 = \epsilon_1 E_1 \;,\; D_2 = \epsilon_2 E_2 \;,\; D_3 = \epsilon_3 E_3 .$$

$E = E_x \hat{x} \Rightarrow D = D_x \hat{x} = \epsilon_1 E_x \hat{x}$.

This is the principal axes. and principal planes of the crystal.

$$\Rightarrow n_1 = \left(\frac{\epsilon_1}{\epsilon_0}\right)^{1/2} \quad n_2 = \left(\frac{\epsilon_2}{\epsilon_0}\right)^{1/2} \quad n_3 = \left(\frac{\epsilon_3}{\epsilon_0}\right)^{1/2}.$$

n_1, n_2, n_3: principal refractive indices (ϵ_0 = permittivity of free space).

Biaxial, Uniaxial, Isotropic Crystals

Isotropic : $n_1 = n_2 = n_3$.
Uniaxial $n_1 = n_2$
Biaxial all different.

Uniaxial. $n_1 = n_2 = n_0 \quad n_3 = n_e$ } notation that is used.

n_0 - ordinary index
n_e = extraordinary index

$n_e > n_0$ (positive uniaxial).
$n_e < n_0$ (negative uniaxial).

z axis in a uniaxial crystal is called the optic axis.

Impermeability Tensor.

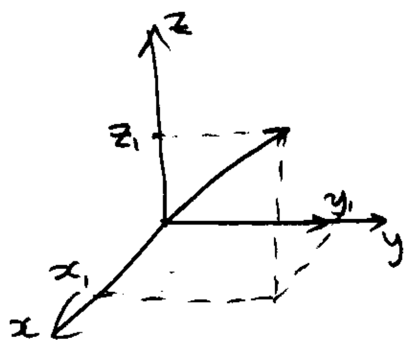
$$\vec{D} = \overleftrightarrow{\epsilon}\vec{E} \Rightarrow \vec{E} = \overleftrightarrow{\epsilon}^{-1}\vec{D}$$

$$\overleftrightarrow{\eta} = \epsilon_0 \overleftrightarrow{\epsilon}^{-1} \text{ : impermeability tensor.}$$

$$\epsilon_0 \vec{E} = \eta \vec{D}$$

Geometrical Representations of vectors and tensors:

Vector.

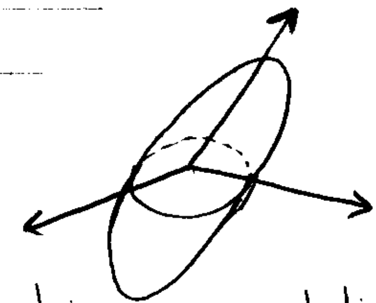


$$\vec{p} = x_1\hat{x} + y_1\hat{y} + z_1\hat{z}.$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$ 3 numbers represent vector.

9 numbers represent tensor.

Symmetrical tensor. – 6 numbers. (ellipsoid) quadratic surface.



$$\sum_{ij} \epsilon_{ij} x_i x_j = 1 .$$

known as quadric representation.

In principal coordinate system: ϵ_{ij} is diagonal:

$$\epsilon_1 x_1^2 + \epsilon_2 x_2^2 + \epsilon_3 x_3^2 = 1 .$$

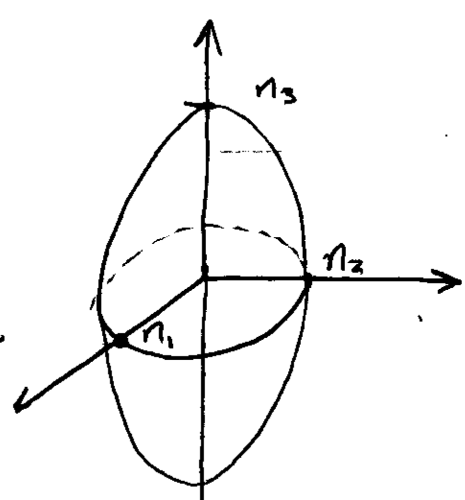
Principal axes are those of the tensor, axes are half-lengths.
$\epsilon_1^{-1/2}$, $\epsilon_2^{-1/2}$, $\epsilon_3^{-1/2}$

<u>Index Ellipsoid</u> (optical indicatrix) $\overleftrightarrow{\eta} = \epsilon_0 \overleftrightarrow{\epsilon}^{-1}$

$$\sum \eta_{ij}\, x_i x_j = 1 .$$

→ Principal axes : $\dfrac{x_1^2}{n_1^2} + \dfrac{x_2^2}{n_2^2} + \dfrac{x_3^2}{n_3^2} = 1.$

$$\eta = \begin{bmatrix} 1/n_1^2 & 0 & 0 \\ 0 & 1/n_2^2 & 0 \\ 0 & 0 & 1/n_3^2 \end{bmatrix} .$$



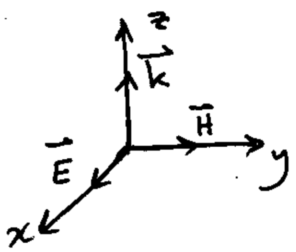
x, y, z are the principal axes
n_1, n_2, n_3 : refractive indices

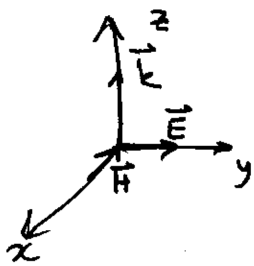
isotropic: sphere uniaxial : $n_1 = n_2$

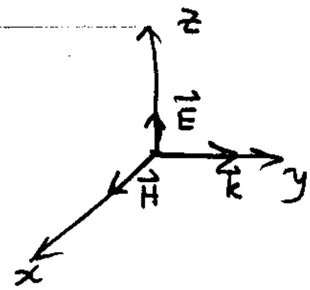
Propagation along a principal axes.

Normal modes.

x-y-z : coordinate system.







$\frac{c_0}{n_1}$ $\qquad$ $\frac{c_0}{n_2}$

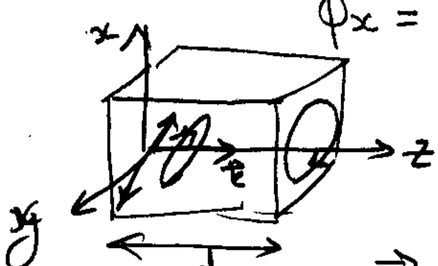
$D_1 = \epsilon_1 E_1 \cdot \quad c^2 = \frac{1}{\sqrt{\mu_0 \epsilon_1}}$

Normal modes for propagat along z-direct are x, y polarized waves.

Polarizat along an arbitrary direction.

- Propagat along $\hat{z}$, arbitrary polarizat in x-y plane.

Analyse as sum of normal modes, $\hat{x}$ polarized mode goes at $\frac{c_0}{n_1}$, $\hat{y}$ polarized at $\frac{c_0}{n_2}$ $\Rightarrow$ phase change.



$$\phi_x = n_1 k_0 d, \quad \phi_y = n_2 k_0 d.$$

$$\phi = \phi_x - \phi_y = (n_2 - n_1) k_0 d.$$

$\Rightarrow$ elliptically polarized light out.

$\Rightarrow$ <u>wave retarder</u>.

Propagation in Arbitrary Direction.

$\hat{u}$ – direction of propagation.

Normal modes are linearly polarized waves, n_a and n_b are the refractive indices and the directions of these modes determined as follows. (see attached picture).

1. Draw plane passing through origin of the index ellipsoid normal to $\hat{u}$. The intersect of this plane with the ellipsoid is called the index ellipse.
2. half-lengths of the index ellipse are n_a & n_b. of the normal modes.
3. Directions of major and minor axes of the index ellipse are directions of the vectors $\vec{D}_a$ and $\vec{D}_b$ for the normal modes (these are orthogonal).
4. $\vec{E}_a$ and $\vec{E}_b$ are determined from $\vec{D}_a$ and $\vec{D}_b$

$$\vec{D}_a = \epsilon_a \vec{E}_a \qquad \vec{D}_b = \epsilon_b \vec{E}_b \qquad n_a = \left(\frac{\epsilon_a}{\epsilon_0}\right)^{1/2}$$

$$n_b = \left(\frac{\epsilon_b}{\epsilon_0}\right)^{1/2}$$

The Dispersion Relation.

$\vec{D} = \overleftrightarrow{\epsilon}\,\vec{E}$ All fields: $\exp(-j\vec{k}\cdot\vec{r})$. $\vec{k} = k\hat{u}$.

$\Rightarrow \vec{k} \times \vec{H} = -\omega \vec{D}$ ($\vec{D}$ · Not necessarily parallel to $\vec{E}$).

$\vec{k} \times \vec{E} = \omega \mu_0 \vec{H}$.

(see picture in page 215)

$\vec{D}$ normal to both $\vec{k}$ & $\vec{H}$

$\vec{H}$ normal to both $\vec{k}$ & $\vec{E}$

$\vec{S} = \frac{1}{2}\vec{E} \times \vec{H}^*$ (direction of power flow) orthogonal to $\vec{E}$ & $\vec{H}$.

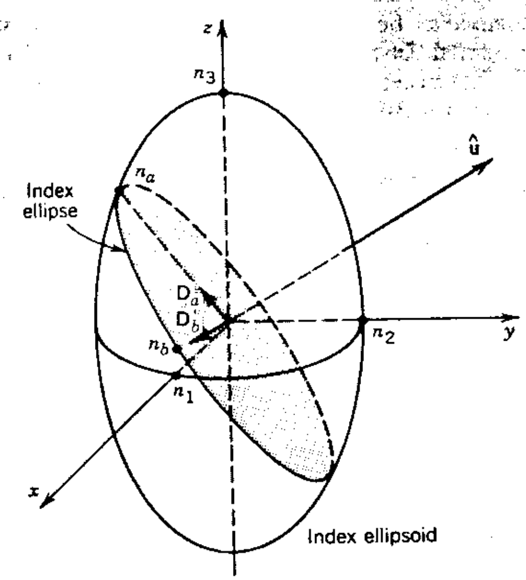


Figure 6.3-6 Normal modes determined from the index ellipsoid.

The Dispersion Relation

To determine the normal modes for a plane wave traveling in the direction $\hat{\mathbf{u}}$, we use Maxwell's equations (5.3-2) to (5.3-5) and the medium equation $\mathbf{D} = \boldsymbol{\epsilon}\mathbf{E}$. Since all fields are assumed to vary with the position $\mathbf{r}$ as $\exp(-j\mathbf{k}\cdot\mathbf{r})$, where $\mathbf{k} = k\hat{\mathbf{u}}$, Maxwell's equations (5.3-2) and (5.3-3) reduce to

$$\mathbf{k} \times \mathbf{H} = -\omega \mathbf{D} \tag{6.3-8}$$

$$\mathbf{k} \times \mathbf{E} = \omega \mu_o \mathbf{H}. \tag{6.3-9}$$

It follows from (6.3-8) that $\mathbf{D}$ is normal to both $\mathbf{k}$ and $\mathbf{H}$. Equation (6.3-9) similarly indicates that $\mathbf{H}$ is normal to both $\mathbf{k}$ and $\mathbf{E}$. These geometrical conditions are illustrated in Fig. 6.3-7, which also shows the Poynting vector $\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^*$ (direction of power

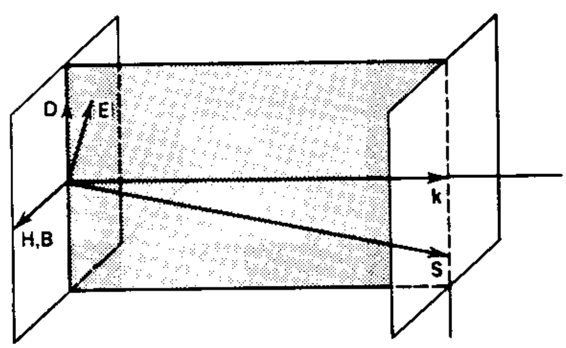


Figure 6.3-7 The vectors **D**, **E**, **k**, and **S** all lie in one plane to which **H** and **B** are normal. $\mathbf{D} \perp \mathbf{k}$ and $\mathbf{E} \perp \mathbf{S}$.

$\vec{D}, \vec{E}, \vec{k}$ and $\vec{S}$ lie in one plane to which $\vec{H}$ and $\vec{B}$ are normal.

In this plane $\vec{D} \perp \vec{k}$ and $\vec{S} \perp \vec{E}$ $\quad \vec{D}$ not necessarily parallel to $\vec{E}$, $\vec{S}$ not necessarily parallel to $\vec{k}$.

$$\vec{k} \times (\vec{k} \times \vec{E}) \frac{1}{\omega \mu_0} + \omega \vec{D} = 0.$$

$$\vec{k} \times (\vec{k} \times \vec{E}) + \omega^2 \mu_0 \overleftrightarrow{\varepsilon} \vec{E} = 0.$$

vector equation, $\vec{E}$ must satisfy. In principal axis:

$$\underbrace{\begin{bmatrix} n_1^2 k_0^2 - k_2^2 - k_3^2 & k_1 k_2 & k_1 k_3 \\ k_2 k_1 & n_2^2 k_0^2 - k_1^2 - k_3^2 & k_2 k_3 \\ k_3 k_1 & k_3 k_2 & n_3^2 k_0^2 - k_1^2 - k_2^2 \end{bmatrix}}_{\overleftrightarrow{A}} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\vec{k} = (k_1, k_2, k_3)$. $\quad k_0 = \frac{\omega}{c_0} \quad (n_1, n_2, n_3)$ – refractive indices.

Nontrivial solution if determinant $\overleftrightarrow{A} = 0$.

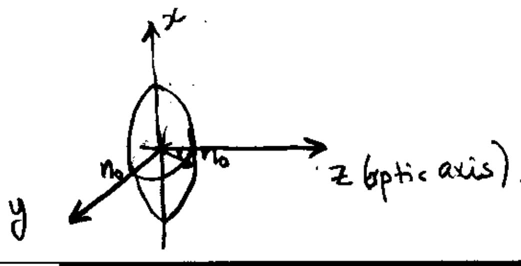
$$\Rightarrow \omega = \underbrace{\omega(k_1, k_2, k_3)}_{\text{nonlinear function}}.$$

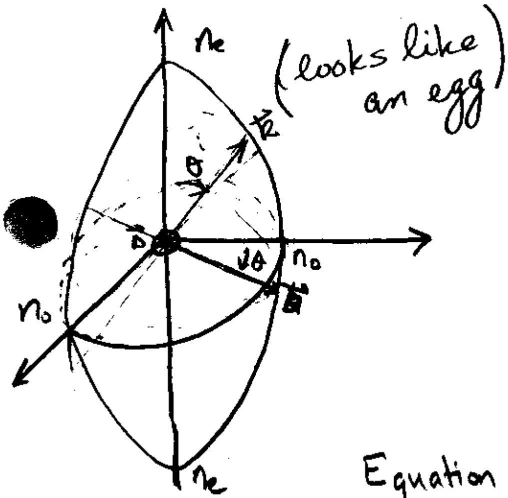
$\omega(k_1, k_2, k_3)$ = dispersion relation is the equation of a surface k_1, k_2, k_3 known as the <u>normal surface</u> or <u>$\vec{k}$ surface</u>.

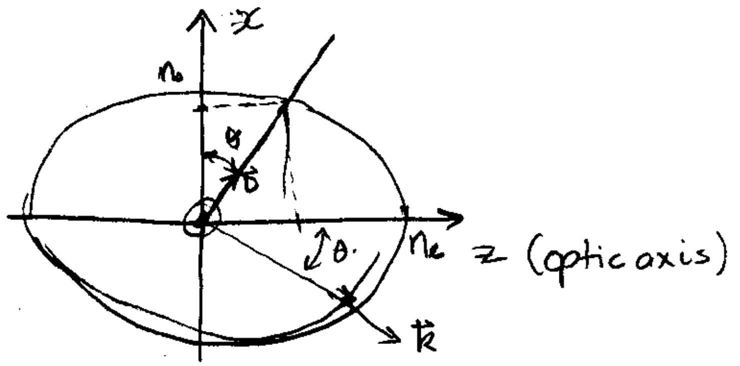
Intersection of $\hat{u}$ with $\vec{k}$ surface determines the vector $\vec{k}$ with $k = \frac{n\omega}{c_0}$.
$\Rightarrow$ index of refraction.

two intersections $\Rightarrow$ two normal modes of each $\hat{u}$.

<u>Example</u>. Special Case: <u>Uniaxial Crystals</u>. $\quad n_1 = n_2 = n_o \quad n_3 = n_e$.
index ellipsoid is an ellipsoid of revolution.



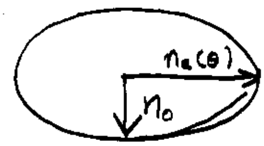




Equation for ellipse : $\dfrac{x^2}{n_o^2} + \dfrac{z^2}{n_e^2} = 1$

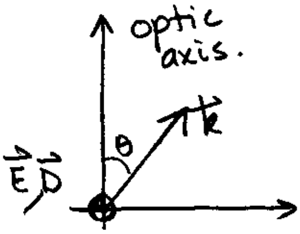
$z = n_e(\theta) \sin\theta \qquad x = n_e(\theta) \cos\theta .$

$$\Rightarrow \quad \frac{\sin^2\theta}{n_e^2} + \frac{\cos^2\theta}{n_o^2} = \frac{1}{n_e^2(\theta)} .$$

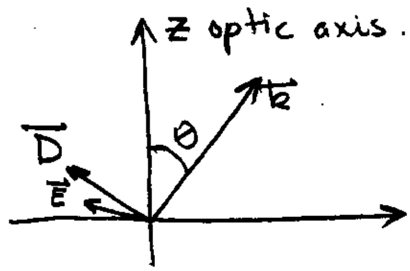


$n_a = n_o \qquad n_b = n(\theta) .$

First is called ordinary ray.
Second is called extraordinary ray.



o-wave



e-wave

$n_e(\theta)$ - refractive index of the extraordinary ray.

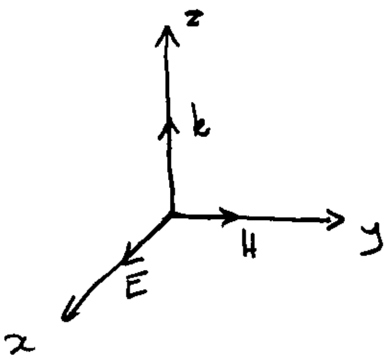
$n_o \leq n_e(\theta) \leq n_e \qquad$ if $n_e > n_o$

$n_o \geq n_e(\theta) \geq n_e \qquad$ if $n_o > n_e$.

Answers to Questions.

① Does phase change ⇒ frequency change?
② What determines color λ or f?
③ $\vec{k} = k_z \hat{z}$ How does this affect E_x & E_y?

Consider the following:



$$\vec{E} = E_x \hat{x} \qquad \vec{k} = k \hat{z}.$$

In anisotropic media

$$D_x = \epsilon_x E_x \qquad \epsilon_x = \epsilon_0 n_x^2$$

$$\text{or } n_x = \left(\frac{\epsilon_x}{\epsilon_0}\right)^{1/2}$$

Maxwells Equation ⇒ Wave Equat.

Assume time harmonic $e = Re\{\vec{E} e^{j\omega t}\}$. $\quad \vec{d} = \epsilon_0 \vec{e} + \vec{p}$

⇒ where $\vec{E} = E_x \hat{x}$. $\quad \vec{D} = \epsilon_x E_x \hat{x}$.

$$\nabla \times \vec{H} = j\omega \vec{D} \qquad \vec{B} = \mu \vec{H}$$

$$\nabla \times \vec{E} = -j\omega \vec{B}$$

$$\nabla \cdot \vec{D} = 0$$

$$\nabla \cdot \vec{B} = 0.$$

$$\nabla \times \nabla \times \vec{E} = -j\omega(\nabla \times \vec{B}) = -\nabla^2 \vec{E} + \nabla(\nabla \cdot \vec{E}).$$

$$\nabla \cdot \vec{E} = 0 \quad \text{(nondispensive, homogenous).}$$

$$\nabla^2 \vec{E} = -j\omega(j\omega\mu)\vec{D} = \omega^2 \mu \vec{D}$$

$\boxed{\nabla^2 \vec{E} = \omega^2 \mu \vec{D}}$ general Equat.

$\vec{D} = \epsilon_x E_x \hat{x} = \epsilon_x \vec{E}_x(\vec{r})$, $\nabla^2 \vec{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \vec{E}_x(\vec{r})$.

$$\nabla^2 \vec{E}(\vec{r}) - \omega^2 \mu \epsilon_x \vec{E}_x(\vec{r}) = 0$$

$$\omega^2 \mu \epsilon_x = \frac{\omega^2 n_x^2}{c_0^2} \qquad c_0 = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \qquad \epsilon_x = \epsilon_0 n_x^2$$

(3) Note it is n_x – the index in the x-direction.

Now – for plane waves: $\vec{E}(\vec{r}) = \vec{E}_0 \exp(-j \vec{k} \cdot \vec{r})$.

$$\Rightarrow \vec{E}_x(\vec{r}) = \underbrace{\vec{E}_x^{\,0}}_{\text{constant vector}} \underbrace{\exp(-j k_z z)}_{\text{spatial dependence}}.$$

$$\Rightarrow \nabla^2 \vec{E} = \nabla^2 E_x(\vec{r}) = k_z^2 \vec{E}_x^{\,0} \exp(-j k_z z).$$

$$\therefore \quad k_z^2 E_x^0 \hat{x} \exp(-jk_z z) - \frac{\omega^2 n_x^2}{c_0^2} \cdot E_x^0 \exp(-jk_z z) \hat{x} = 0$$

$$\Rightarrow \boxed{k_z = \frac{\omega n_x}{c_0}}$$

What if $\vec{E} = \left(E_x^0 \exp(-jk_{zx} z) \hat{x} + E_y^0 \exp(-jk_{zy} z) \hat{y} \right)$. $\vec{D} = \epsilon_x E_x \hat{x} + \epsilon_y E_y \hat{y}$

we would get $\exp(\cancel{-jk_z z}) \left(k_{zx}^2 E_x^0 \hat{x} + k_{zy}^2 E_y^0 \hat{y} \right)$

$$- \frac{\omega^2}{c_0^2} \left(n_x^2 E_x^0 \hat{x} + E_y^0 n_y^2 \hat{y} \right) \exp(\cancel{-jk_z z}).$$

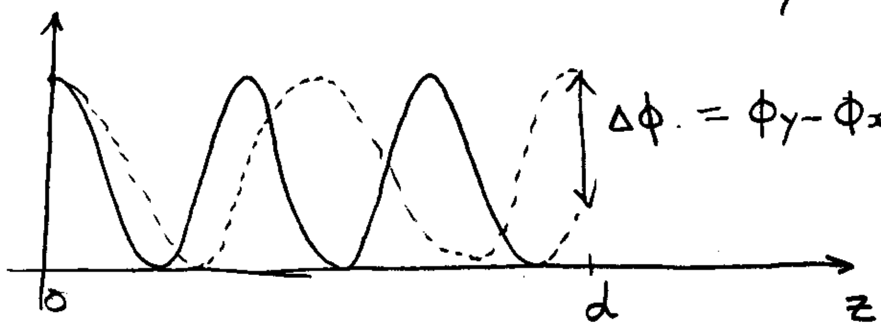
$$\Rightarrow \left(k_{zx}^2 \hat{x} + k_{zy}^2 \hat{y} \right) - \frac{\omega^2 n_x^2}{c_0^2} \hat{x} - \frac{\omega^2}{c_0^2} n_y^2 \hat{y} = 0$$

$$k_{zx} = \frac{\omega n_x}{c_0} \qquad k_{zy} = \frac{\omega}{c_0} n_y.$$

Because of the anisotropic behavior of the medium we have two different values of k in the x-direction and the y-direction.

$$k = \frac{2\pi n}{\lambda_0} \quad \Rightarrow \quad \lambda = \frac{\lambda_0}{n}$$

$$\lambda_x = \frac{\lambda_0}{n_x} \qquad \lambda_y = \frac{\lambda_0}{n_y}$$



$$\Delta\phi = \phi_y - \phi_x = (n_y - n_x) k_0 d.$$

Solution $$\vec{E} = E_x^0 \cos(\omega t - k_{zx} z)\,\hat{x} + E_y^0 \cos(\omega t - k_{zy} z)\,\hat{y}$$

polarization $$\begin{bmatrix} E_x^0 \\ E_y^0 e^{j\Delta\phi} \end{bmatrix}$$

phase velocity $\omega t - k_{zx} z$ = constant phase.

$$\frac{d}{dt}(\omega t - k_{zx} z) = 0 \Rightarrow \omega - k_{zx}\frac{dz}{dt} = 0.$$

or $$\frac{dz}{dt} = \frac{\omega}{k_{zx}} \qquad \boxed{k_{zx} = \frac{2\pi n_x}{\lambda_0}}$$

$k_{zy} = \frac{2\pi n_y}{\lambda_0}$ $\Rightarrow$ different phase velocities.

② Color determined by frequency. ω.

λ in vacuum is generally quoted for wavelengths of colors.

not that λ changes but frequency ω stays the same.

$$\cos\left(\omega t - \frac{2\pi n_x z}{\lambda_0}\right). \quad \cos\left(\omega t - \frac{2\pi n_y z}{\lambda_0}\right).$$

(1) At $z = z_0$ (a constant the amplitude of both cos terms change with ω.