Electro-optics of Anisotropic Media.

Pockels & Kerr Effects.

$\overline{E} = (E_1, E_2, E_3)$ is applied to a crystal
$\Rightarrow \overleftrightarrow{\eta}$ permeability tensor is altered.

$\overleftrightarrow{\eta} = \epsilon_0 \overleftrightarrow{\epsilon}^{-1}$ $\quad \eta_{ij} = \eta_{ji}$.

Apply $\vec{E} \Rightarrow \eta_{ij} = \eta_{ij}(\vec{E})$.

$$\eta_{ij}(\vec{E}) = \eta_{ij} + \sum_k r_{ijk} E_k + \sum_{kl} s_{ijkl} E_k E_l \;, \quad i,j,k,l = 1,2,3\ldots$$

$$\eta_{ij} = \eta_{ij}(0) \;;\; r_{ijk} = \left.\frac{\partial \eta_{ij}}{\partial E_k}\right|_{\vec{E}=0} \quad s_{ijkl} = \frac{1}{2} \left.\frac{\partial^2 \eta_{ij}}{\partial E_k \partial E_l}\right|_{\vec{E}=0}.$$

$\rightarrow$ generalized form of $\eta(E) - \eta + rE + sE^2$.

r replaced by $3^3 = 27$ coefficients $\{r_{ijk}\}$.

s replaced by $3^4 = 81$ coefficients $\{s_{ijkl}\}$.

r_{ijk} - Pockels coefficients - linear electro-optic. 3^{rd} rank tensor
s_{ijkl} - Kerr coefficients - quadratic electro-optic coeff. 4^{th} rank tensor

Symmetry. $\overleftrightarrow{\eta}$ - symmetric $\quad r_{ijk}$ s_{ijkl} invariant under permutations of i and j.

$r_{ijk} = r_{jik} \qquad s_{ijkl} = s_{jikl}$

in addition: $\qquad s_{ijkl} = s_{ijlk}$

ij permutations $\Rightarrow$ 6 elements eg: 11, 12, 13, 22, 23, 33

$kl \Rightarrow$ 6 elements. $\rightarrow r_{ijk}$ has 6×3 elements

s_{ijkl} : 6×6 elements.

We rename $(i,j), i,j = 1,2,3 \rightarrow$ single index $I = 1,2,3\cdots,6$

$r_{ijk} \rightarrow r_{IK}$. $s_{ijkl} \rightarrow s_{IK}$. $K = 1,2,3\cdots 6$.

example $r_{12K} \rightarrow r_{6K}$ $s_{1231} \rightarrow s_{65}$

From Table:

j	$i=1$	2	3
1	1	6	5
2	6	2	4
3	5	4	3

$(i,j) = (3,2) \rightarrow I=4$.

r (3rd rank tensor) $\rightarrow$ 6X3 matrix

s (4th rank tensor) $\rightarrow$ 6X6 matrix.

Crystal symmetry.

Crystal symmetry imposes more constraints on r & s

$\Rightarrow$ some entries must be equal, some zero, opposite in sign or some other relat.

Example Pockels Effect.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ 0 & r_{41} & 0 \\ 0 & 0 & r_{41} \end{bmatrix} \qquad \begin{bmatrix} & & \\ & & \\ & & \\ r_{41} & & \\ & r_{41} & \\ & & r_{63} \end{bmatrix} \qquad \begin{bmatrix} 0 & -r_{22} & r_{13} \\ 0 & r_{22} & r_{13} \\ 0 & 0 & r_{33} \\ 0 & r_{51} & 0 \\ r_{51} & 0 & 0 \\ -r_{22} & 0 & 0 \end{bmatrix}$$

Cubic $\bar{4}3m$ tetragonal $\bar{4}2m$ Trigonal $3m$

GaAs, CdTe, InAs. (KDP, ADP) ($LiNbO_3$, $LiTaO_3$).

Kerr Coefficients. S_{jk} for an isotropic medium.

$$\begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{21} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{44} \end{bmatrix} \qquad S_{44} = \frac{S_{11} - S_{12}}{2}.$$

Pockels Effect. $\vec{E} = (E_1, E_2, E_3)$

Determine optical properties.

① Find principal axes and principal refractive indices n_1, n_2 and n_3 in absence of $\vec{E}$

② Find $\{r_{ijk}\}$ by using r_{Ik} along with contraction rule relating i, j to I.

③ Determine impermeability tensor using.

$$\eta_{ij}(\vec{E}) = \eta_{ij}(0) + \sum_k r_{ijk} E_k$$

where $\eta_{ij}(0)$ is a diagonal matrix with elements $\frac{1}{n_1^2}, \frac{1}{n_2^2}, \frac{1}{n_3^2}$

④ Write equation for modified index ellipsoid:

$$\sum_{i,j} \eta_{ij}(\vec{E})\, x_i x_j = 1$$

⑤ Determine principal axes of modified index ellipsoid by diagonalizing the matrix $\eta_{ij}(\vec{E})$ and find the corresponding principal refractive indices $n_1(\vec{E})$, $n_2(E)$ & $n_3(E)$.

⑥ Given direction of light propagation $\vec{k}$, find the normal modes and their associated refractive indices using the index ellipsoid.

Index ellipsoid

$$\overleftrightarrow{\eta} = \begin{bmatrix} 1/n_1^2 & 0 & 0 \\ 0 & 1/n_2^2 & 0 \\ 0 & 0 & 1/n_3^2 \end{bmatrix}.$$

represent graphically as.

$$\sum \eta_{ij} x_i x_j = 1.$$

Write as equation : $\eta_{11} x_1^2 + \eta_{12} x_1 x_2 + \eta_{13} x_1 x_3 + \eta_{21} x_1 x_2 +$
$\eta_{22} x_2^2 + \eta_{23} x_2 x_3 + \eta_{31} x_3 x_1 + \eta_{32} x_3 x_2$
$+ \eta_{33} x_3^2 = 1$

but $\eta_{11} = \frac{1}{n_1^2}$, $\eta_{22} = \frac{1}{n_2^2}$, $\eta_{33} = \frac{1}{n_3^2}$ all others zero.

$$\Rightarrow \quad \frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} = 1.$$

or

$$\boxed{\frac{x^2}{n_1^2} + \frac{y^2}{n_2^2} + \frac{z^2}{n_3^2} = 1.}$$

equation for ellipsoid !

n_1, n_2, n_3 are refractive indices.

Pockels Effect in anisotropic media:

$$\eta_{ij} = \eta_{ij}(0) + \sum_k r_{ijk} E_k.$$

$$\begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{bmatrix} = \begin{bmatrix} 1/n_1^2 & 0 & 0 \\ 0 & 1/n_2^2 & 0 \\ 0 & 0 & 1/n_3^2 \end{bmatrix} + \sum_k r_{ijk} E_k.$$

Review of concepts - clarifications.

TEM plane waves: $\vec{E}(z,t) = \mathrm{Re}\left\{\vec{A}\,\underbrace{\exp\left[j2\pi\nu\left(t-\frac{z}{c}\right)\right]}_{\text{we know always there.}}\right\}$

and $\vec{A} = A_x\hat{x} + A_y\hat{y}$ $\quad A_x$ & A_y are complex.

Jones vector: $J = \begin{bmatrix} A_x \\ A_y \end{bmatrix}$ completely describes wave!

$\vec{D} = \epsilon\vec{E} + \vec{P}$ $\qquad D_i = \sum_j \epsilon_{ij} E_j$ anisotropic medium.

or $\vec{D} = \overleftrightarrow{\epsilon} E$.

example.

$$D_x = \epsilon_{xx}E_x + \epsilon_{xy}E_y + \epsilon_{xz}E_z$$
$$D_y = \epsilon_{yx}E_x + \epsilon_{yy}E_y + \epsilon_{yz}E_z$$
$$D_z = \epsilon_{zx}E_x + \epsilon_{zy}E_y + \epsilon_{zz}E_z$$

or $\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \underbrace{\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yz} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}}_{\text{permittivity tensor.}} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$.

Rewrite $\overleftrightarrow{\eta} = \epsilon_0 \overleftrightarrow{\epsilon}^{-1}$: impermeability tensor.

$\boxed{\vec{E} = \frac{\overleftrightarrow{\eta}}{\epsilon_0}\vec{D}}$. o.k.? or $\vec{E} = \frac{1}{\epsilon_0}\sum_j \eta_{ij} D_j$

In addition: we define $n_1 = \left(\frac{\epsilon_1}{\epsilon_0}\right)^{1/2}$, $n_2 = \left(\frac{\epsilon_2}{\epsilon_0}\right)^{1/2}$, $n_3\left(\frac{\epsilon_3}{\epsilon_0}\right)^{1/2}$

$\sum_k r_{ijk} E_k = ?$ $\quad i, j, k = 1, 2, 3.$ $\quad$ 27 coefficients.

How many total elements. i, j are indices $\Rightarrow$ 9 final values.

$$
\begin{array}{l}
\\
b_{21} \leftarrow
\end{array}
\left[
\begin{array}{lll}
r_{111}E_1 + r_{112}E_2 + r_{113}E_3, & \overbrace{r_{121}E_1 + r_{122}E_2 + r_{123}E_3}^{b_{12}}, & r_{131}E_1 + r_{132}E_2 + r_{133}E_3 \\
r_{211}E_1 + r_{212}E_2 + r_{213}E_3, & r_{221}E_1 + r_{222}E_2 + r_{223}E_3, & \ldots \\
 & &
\end{array}
\right]
$$

But from symmetry: $r_{ijk} = r_{jik}$.

$\Rightarrow \quad r_{121} = r_{211}$

$$\therefore \quad b_{12} = r_{121}E_1 + r_{122}E_2 + r_{123}E_3 = r_{211}E_1 + r_{212}E_2 + r_{213}E_3 = b_{21}$$

<u>Trigonal 3m Crystals.</u> $n_1 = n_2 = n_o$, $n_3 = n_e$ r given earlier.

$\vec{E} = (0, 0, E)$ Electric field along optic axis.

$$\eta_{ij}(E_z) = \eta_{ij}(0) + r_{ijz} E_z .$$

$$\eta_I(E_z) = \eta_I(0) + r_{Iz} E_z . \qquad z = 3.$$

only $I = 1, 2, 3$ are none zero r_{13}, r_{33}. and $r_{23} = r_{13}$.

$$\eta_1 = \eta_1(0) + r_{13} E_3 .$$
$$\eta_3 = \eta_3(0) + r_{33} E_3 .$$
$$\eta_2 = \eta_2(0) + r_{23} E_3 . \quad (r_{23} = r_{13})$$

$$\eta_1 = \frac{1}{n_1^2} \qquad \eta_3 = \frac{1}{n_3^2}$$

$$\frac{1}{n_3^2(E)} = \frac{1}{n_3^2} + r_{33} E_3 \qquad \frac{1}{n_1^2(E)} = \frac{1}{n_1^2} + r_{13} E_3$$

$$\left(\frac{1}{n_o^2} + r_{13} E\right)(x_1^2 + x_2^2) + \left(\frac{1}{n_e^2} + r_{33} E\right) x_3^2 = 1 .$$

$$\frac{1}{n_o^2(E)} = \frac{1}{n_o^2} + r_{13} E$$

$$\frac{1}{n_e^2(E)} = \frac{1}{n_e^2} + r_{33} E$$

$r_{13} E$ & $r_{33} E$ are generally small $\Rightarrow (1+\Delta)^{-1/2} \simeq 1 - \frac{1}{2}\Delta$. $|\Delta|$ small.

$$\Rightarrow \quad n_o(E) \simeq n_o - \frac{1}{2} n_o^3 r_{13} E$$

$$n_e(E) \simeq n_e - \frac{1}{2} n_e^3 r_{33} E$$

crystal remains uniaxial with same principal axes.

Tetragonal $\bar{4}2m$ Crystals (KDP & ADP).

r_{41}, $r_{52} = r_{41}$, r_{63} only nonzero elements.

$\vec{E} = (0, 0, E)$.

$$\eta_4(E) = \eta_4(0) + r_{41} E_1 = \eta_4(0) = \eta_{32}(0) = \eta_{23}(0) = 0$$
$$\eta_5(E) = \eta_5(0) + r_{51} E_1 = \eta_5(0) = \eta_{31}(0) = \eta_{23}(0) = 0$$
$$\eta_6(E) = \eta_6(0) + r_{63} E_3 = \eta_6(0) + r_{63} E = r_{63} E$$
$$\eta_{12}(E) = \eta_{12}(0) + r_{63} E = r_{63} E$$

$$\eta_{21}(E) = \eta_{21}(0) + r_{63}E = r_{63}E$$

$$\eta_1(E) = \frac{1}{n_1^2} = \frac{1}{n_o^2}$$

$$\eta_2(E) = \frac{1}{n_2^2} = \frac{1}{n_o^2}$$

$$\eta_3(E) = \frac{1}{n_3^2}.$$

$$\Rightarrow \quad \frac{x_1^2 + x_2^2}{n_o^2} + \frac{x_3^2}{n_e^2} + 2r_{63}E\,x_1x_2 = 1.$$

Can be rewritten as (by diagonalizing $\overleftrightarrow{\eta}$ matrix).

$$\frac{u_1^2}{n_1^2(E)} + \frac{u_2^2}{n_2^2(E)} + \frac{u_3^2}{n_3^2(E)} = 1.$$

$$\left.\begin{aligned} \frac{1}{n_1^2(E)} &= \frac{1}{n_o^2} + r_{63}E \\ \frac{1}{n_2^2(E)} &= \frac{1}{n_o^2} - r_{63}E \\ n_3(E) &= n_e. \end{aligned}\right\}$$ uniaxial crystal $\Rightarrow$ biaxial crystal.

$$(1+\Delta)^{-1/2} \simeq 1 - \frac{1}{2}\Delta \quad \Rightarrow \quad \begin{aligned} n_1(E) &= n_o - \frac{1}{2}n_o^3 r_{63}E \\ n_2(E) &= n_o + \frac{1}{2}n_o^3 r_{63}E \\ n_3(E) &= n_e. \end{aligned}$$

Cubic Crystals $\bar{4}3m$. (GaAs, CdTe, InAs). $n_1 = n_2 = n_3 = n$ isotropic. $r_{41} = r_{52} = r_{63}$.

$$\vec{E} = (0,0,E).$$

$$\begin{aligned} \eta_1 &= \eta_1(0) & \eta_4 &= r_{41}E_1 = 0 \\ \eta_2 &= \eta_2(0) & \eta_5 &= r_{52}E_2 = 0 \\ \eta_3 &= \eta_3(0) & \eta_6 &= r_{63}E_3 = r_{41}E \end{aligned}$$

$$\Rightarrow \eta_{21} = \eta_{12} = r_{41}E$$

$$\frac{x_1^2 + x_2^2 + x_3^2}{n^2} + 2 r_{41} E x_1 x_2 = 1$$

new axes rotated by $45°$ about z axis

$$\left.\begin{aligned} n_1(E) &\simeq n - \tfrac{1}{2} n^3 r_{41} E \\ n_2(E) &\simeq n + \tfrac{1}{2} n^3 r_{41} E \\ n_3(E) &\simeq n \end{aligned}\right\}$$

biaxial crystal. ⇒ uniaxial ⟶ biaxial.

Notice in all of the above, the crystals principal axes do not change as the electric field E increases

⇒ Normal modes stay the same, refractive indices change.

⇒ Crystals can be used as a phase modulator, wave retarder or intensity modulator.

<u>Kerr Effect</u> $\quad \eta_{ij}(E) = \eta_{ij}(0) + \sum_{kl} s_{ijkl} E_k E_l$.

<u>Isotropic medium</u>: $\quad \vec{E} = (0,0,E)$

$$\left(\frac{1}{n^2} + s_{12}E^2\right)(x_1^2 + x_2^2) + \left(\frac{1}{n^2} + s_{11}E^2\right)x_3^2 = 1.$$

$$\frac{1}{n_o^2(E)} \simeq \frac{1}{n^2} + s_{12}E^2$$

$$\frac{1}{n_e^2(E)} = \frac{1}{n^2} + s_{11}E^2$$

or

$$\left.\begin{aligned} n_o(E) &\simeq n - \tfrac{1}{2} n^3 s_{12} E^2 \\ n_e(E) &\simeq n - \tfrac{1}{2} n^3 s_{11} E^2. \end{aligned}\right\}$$

quadratic decreasing functions of E.

Modulators from Pockels cells - can also be Kerr cells.

Phase modulators. $n(E) \simeq n - \frac{1}{2} r n^3 E$ n and r

refractive index — Pockels coefficient

$$E = \frac{V}{d} \qquad \phi = \phi_0 - \pi \frac{V}{V_\pi} \quad \text{where } \phi_0 = \frac{2\pi n L}{\lambda_0}$$

$V_\pi = \frac{d}{L} \frac{\lambda_0}{r n^3}$ ÷ half wave voltage. n and r can be determined as follows.

Trigonal 3m Crystal. $\vec{E} = (0,0,E)$. $\Rightarrow$ stays uniaxial.

$$n_o(E) \simeq n_o - \frac{1}{2} n_o^3 r_{13} E$$

$$n_e(E) \simeq n_e - \frac{1}{2} n_e^2 r_{33} E$$

Longitudinal modulator: $\vec{k} = k \hat{z} = (0,0,k)$ parallel to E

$\Rightarrow n = n_o \quad r = r_{13}, \quad d = L$

$LiNbO_3$ $r_{13} = 9.6\,pm/V$, $n_o = 2.3$, $\lambda_0 = 633\,nm$ (HeNe Laser).

$V_\pi = 5.41\,kV$, $5.41\,kV$ is required for π phase change.

Transverse modulator. $\vec{k} = (k,0,0)$ x-direction.

and is polarized in z-direct

$\Rightarrow n = n_e \quad r = r_{33}. \quad d \neq L.$

$LiNbO_3$ $r_{33} = 30.9\,pm/V$ $n_e = 2.2$, $V_\pi = 1.9 \left(\frac{d}{L}\right) \cdot kV$.

if $\frac{d}{L} = 0.1 \Rightarrow V_\pi \simeq 190V$ - much smaller than longitudinal case.

Intensity Modulator

$$\Gamma = \Gamma_0 - \pi \frac{V}{V_\pi} \qquad \Gamma_0 = \frac{2\pi (n_1 - n_2) L}{\lambda_0}$$

$$V_\pi = \frac{(d/L)\lambda_0}{r_1 n_1^3 - r_2 n_2^3}$$

Cell between crossed polarizers $\Rightarrow$ intensity modulator.

Example. Tetragonal $\bar{4}2m$ Crystal (KDP & ADP).

$$\left.\begin{aligned} \vec{E} &= (0,0,E) \\ \vec{k} &= (0,0,k) \end{aligned}\right\} \Rightarrow \text{Normal modes}$$

$n_1 = n_2 = n_0 \qquad r_1 = r_{63}$

$r_2 = -r_{63}$.

recall
$$\begin{cases} n_1(E) = n_0 - \frac{1}{2} n_0^3 r_{63} E \\ n_2(E) = n_0 + \frac{1}{2} n_0^3 r_{63} E \\ n_3(E) = n_e \end{cases}$$

If $d/L = 1$ with $n_1 = n_2 = n_0 \Rightarrow \Gamma_0 = 0$.

$$V_\pi = \frac{\lambda_0}{2 r_{63} n_0^3}$$

KDP at $\lambda_0 = 633\,\text{nm}$
$V_\pi = 8.4\,\text{kV}$.